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A UNIQUENESS CONDITION FOR NONTRIVIAL PERIODIC SOLUTIONS TO THE LIÉNARD EQUATION

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SANTA MONICA • CALIFORNIA

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PREFACE

Part of the Project RAND research program consists of basic supporting studies in mathematics. The mathematical research presented here concerns the periodic solutions of the Liénard differential equation, which applies to a wide variety of physical problems.

SUMMARY

This Memorandum presents a new condition which implies that the differential equation

$$\ddot{x} + f(x)\dot{x} + q(x)x = 0$$

has an essentially unique non-trivial periodic solution, to which all other solutions tend as $t \to \infty$.

ACKNOWLEDGMENT

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A UNIQUENESS CONDITION FOR NONTRIVIAL PERIODIC SOLUTIONS TO THE LIÉNARD EQUATION

The existence and uniqueness of periodic solutions to the equation

$$(1) \quad \ddot{x} + f(x)\dot{x} + q(x)x = 0 \quad (f(x),\ q(x) \text{ continuous})$$

have been widely discussed during the past thirty-five years, notably by Liénard [5] and by Levinson and Smith [4]. The purpose of the present paper is to present a uniqueness condition for periodic solutions which includes one of those given in the latter paper [4] as a special case.

Define the following functions:

$$F(x) = \int_0^x f(x)dx,$$

$$Q(x) = \int_0^x q(x)xdx.$$

In the Poincaré phase-plane (i.e., the $(x, \dot{x})$-plane), define (as in [4]) a pseudo-energy function

$$E(x, \dot{x}) = \tfrac{1}{2}(\dot{x} + F(x))^2 + Q(x).$$

Using this energy function, the following result may be proved:

Lemma. Suppose the following conditions hold on the coefficients of equation (1):

(a) $q(x) > 0$ for all $x \neq 0$;

(b) there exist real numbers $a < 0$ and $b > 0$ such that $F(a) = F(b) = F(0) = 0$, and $xF(x) < 0$ for all other x such that $a \leq x \leq b$;

(c) $f(x) \geq 0$ for $x < a$ and $x > b$;

(d) $\lim_{x\to\infty} Q(x) = \lim_{x\to\infty} Q(-x) = \infty$.

Then there exists at least one nontrivial periodic solution, and at most one which enters both the region $x \leq a$ and the region $x \geq b$.

Proof. Along solution curves, $dE/dt = -xq(x)F(x)$. Thus the phase-plane is divided into three strips (see Fig. 1): $x < a$ (where $dE/dt < 0$), $a \leq x \leq b$ (where $dE/dt \geq 0$), and $x > b$ (where $dE/dt < 0$). Now let Γ_0 be a limit cycle which enters all three regions, and suppose it passes through the point $(0, \dot{x}_0)$. Its journey around the phase-plane naturally divides into five parts (as in the figure).

Now say $x_A > x_0$. Let Γ_A be the curve through $(0, x_A)$. In part I of Γ_A's journey about the phase-plane, it is gaining energy. Since, however, $\dot{x}_{\Gamma_A} > \dot{x}_{\Gamma_0}$ for corresponding values of x in part I, it follows that the change in E per unit x is less on Γ_A than on Γ_0 (for corresponding x). Thus the amount of energy gained by

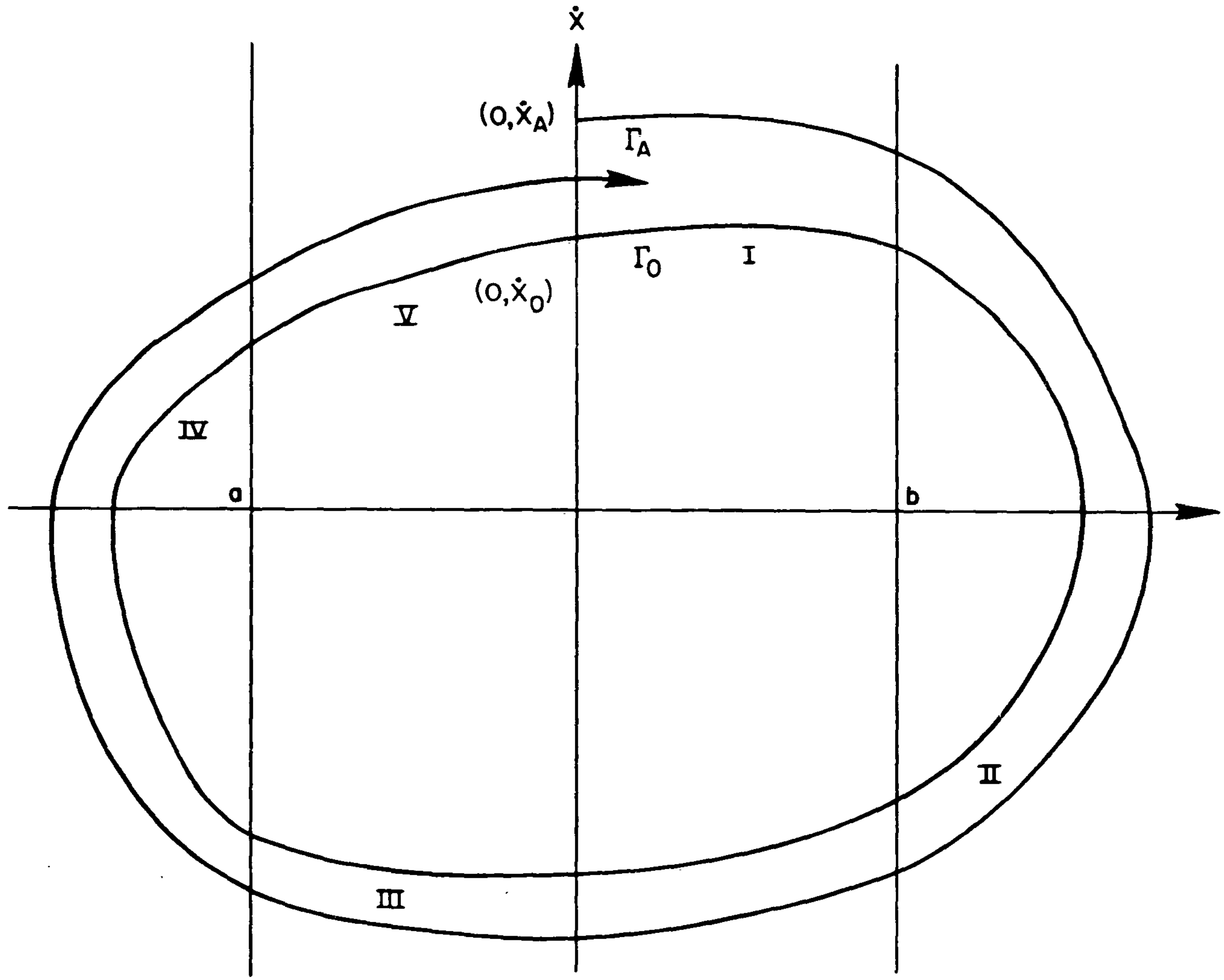

Fig. 1 —The proof of the lemma

going from $x = 0$ to $x = b$ along Γ_A is less than that gained by going along Γ_0. Similarly, Γ_A gains less energy than Γ_0 in parts III and V. A simple computation ([1], [2]) shows that Γ_A loses more energy than Γ_0 in parts II and IV. Thus Γ_A must, on balance, lose energy in making a single circuit.

Similarly any Γ_B which starts inside Γ_0 will gain energy in making a circuit provided it enters both the left-hand and right-hand regions. Thus, if such a limit cycle exists, it is unique and stable on both sides.

To show a limit cycle exists, we need only find an $\dot{x}_B$ so small that the cycle starting at $(0, \dot{x}_B)$ gains energy, and an $\dot{x}_A$ so large that the cycle starting at $(0, \dot{x}_A)$ loses energy. This is easy. Note, however, that such a limit cycle need not enter all three regions (as we assumed Γ_0 did [1], [3]), and thus we cannot conclude that there is only one limit cycle unless we insure that any limit cycle will enter both the region $x < a$ and the region $x > b$. Levinson and Smith met this problem by assuming $f(x)$ and $q(x)$ symmetric about zero, but we shall show that a much weaker condition is adequate.

<u>Theorem</u> 1. <u>Suppose an equation satisfying the conditions of the lemma satisfies also the following</u>:

$$(*) \int_0^a q(x)x\,dx = \int_0^b q(x)x\,dx.$$

Then there exists a unique (up to translations in t) non-trivial periodic solution, to which all other solutions tend.

Proof. We have already shown that there is at most one limit cycle which enters both the regions $x < a$ and $x > b$. We have also shown that some limit cycle exists. Now we will show that under condition (*), any limit cycle enters both the above regions, and thus uniqueness follows. First note (see Fig. 2) that along the curve $v = -F(x)$,

$$E(x, v) = \int_0^x q(x)x \, dx.$$

Since $\int_0^x q(x)x \, dx$ is monotone decreasing for $x < 0$, and monotone increasing for $x > 0$, it follows that the energy along the curve $v = -F(x)$, $a < x < b$, is always less than

$$\int_0^a q(x)x \, dx = \int_0^b q(x)x \, dx.$$

Now, consider the orbit I (which starts at $(a,0)$) or the orbit II (which starts at $(b, 0)$). The energy E is increasing along these orbits, and thus neither of them can intersect $v = -F(x)$ between a and b, since each orbit has initial energy $\int_0^a q(x)x \, dx$. Thus, I must enter $x > b$ and II must enter $x < a$. It immediately follows that any limit cycle must do likewise, and the proof is complete.

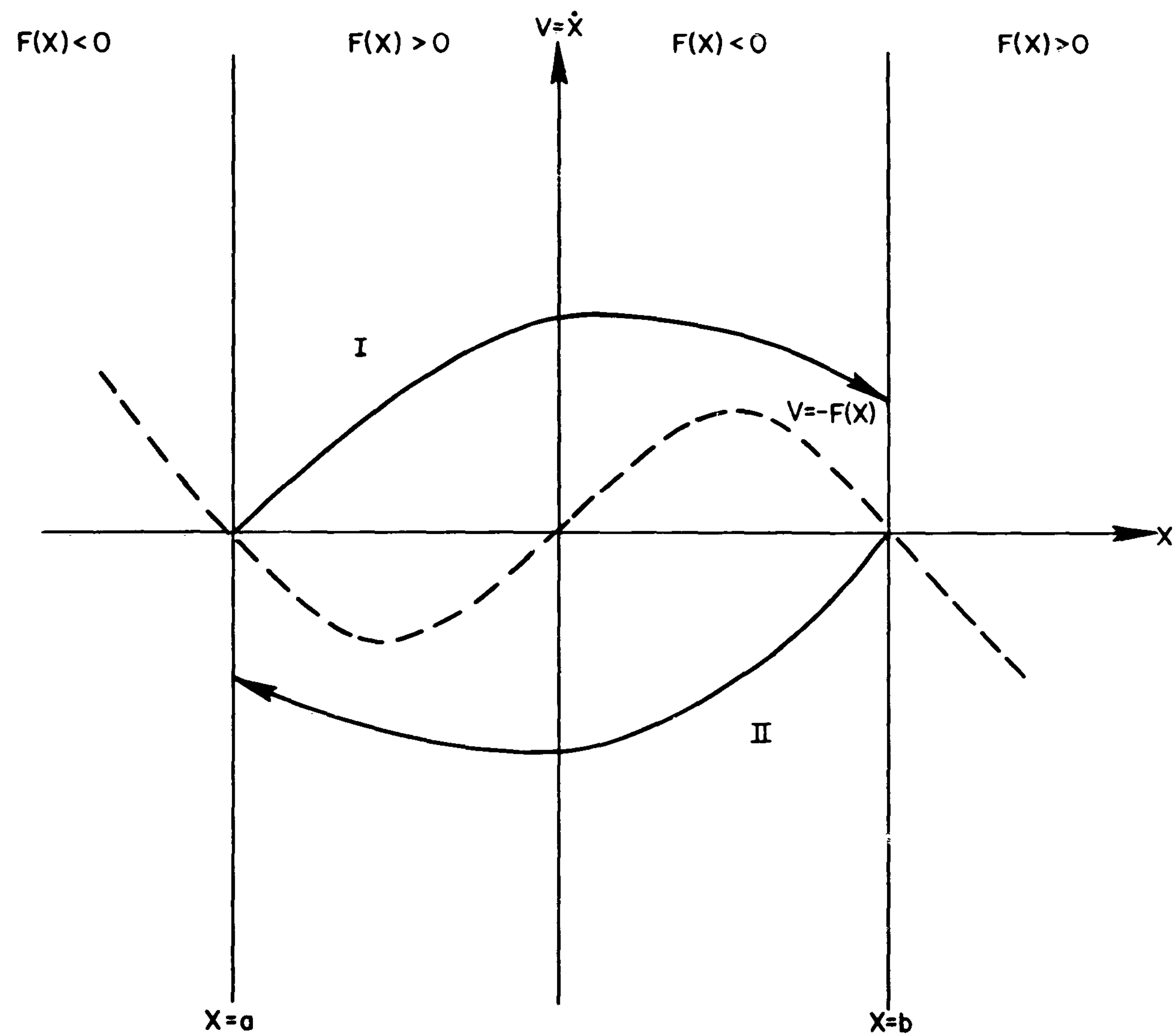

Fig. 2—The proof of the theorem

<u>Corollary</u>. <u>Given an equation of the form</u> (1) <u>which satisfies the following conditions</u>:

(a) $f(x) = f(-x)$, $q(x) = q(-x)$ <u>for all</u> x;

(b) $q(x)x$ <u>is differentiable for all</u> x, <u>and</u> $q(x) > 0$ for $x \neq 0$;

(c) <u>there exists an</u> $a > 0$ <u>such that</u> $f(x) > 0$ <u>for</u> $x > a$;

(d) $F(x) < 0$ <u>for</u> $0 < x < a$, $F(a) = 0$;

(e) $\lim_{x\to\infty} F(x) = \lim_{x\to\infty} Q(x) = +\infty$.

<u>Then there exists a unique (up to translations in</u> t) <u>non-trivial periodic solution</u>.

<u>Proof</u>. Assumption (a) shows that $F(a) = F(-a)$ and $Q(a) = Q(-a)$, and thus the corollary is easily seen to follow. This corollary is the theorem (unnumbered) which Levinson and Smith prove in Section 4 of [4].

We conclude by giving a specific example of an equation which has a unique stable limit cycle by our theorem, but about which the Levinson-Smith theorems are silent:

$$\ddot{x} + (3x^2-4x-3)\dot{x} + q(x)x = 0$$

$$\text{where } q(x) = \begin{cases} -27x \text{ for } x \leq 0 \\ x \text{ for } x > 0 \end{cases}$$

(here $a = -1$, $b = 3$).

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